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EFFECT OF TEMPERATURE SENSITIVITY AND INELASTIC BEHAVIOR OF PHASE MATERIALS ON THE BEARING CAPACITY OF PLANE STRUCTURES WITH UNIFORMLY STRESSED REINFORCEMENT

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An inelastic problem of uniformly stressed reinforcement of plane temperature-sensitive composite structures is formulated. Analytical solutions are obtained for the thermoelastic and inelastic cases. On the basis of these solutions, it is shown that the bearing capacity for inelastic projects can be increased severalfold as compared to thermoelastic projects, and reinforcement can be substantially saved in the inelastic case under fixed loading. Despite the worsening of strength characteristics of the composition phases, the bearing capacity of the structure remains almost unchanged upon heating in the inelastic case and can even increase in the thermoelastic case.

Key words: *composites, uniformly stressed reinforcement, temperature sensitivity, termoelasticity, thermoplasticity, uniform deformation.*

One of the strength criteria in rational design of composite structures under static loading is the uniform stress of fibers along their trajectories, which allows one to use the bearing capacity of high-strength reinforcement most completely and create reliable structures even with a low strength of the binder. Many papers deal with uniformly stressed reinforcement or rational reinforcement (RR) (see, e.g., [1–8]). Until now, however, the study of the RR problem either employed the fiber (grid) model of the reinforced layer [7, 8], which ignores the mechanical behavior of the binder and, hence, the effect of the thermal or radiative action on the structure [5], or the behavior of all phases of the composition was assumed to be linearly elastic, i.e., the real behavior of phase materials beyond the yield point was neglected. The efficiency of using the bearing capacity of real fibers was not estimated in considering the RR problem in the elastic formulation. In addition, it is known that the physicomechanical properties of many materials used to prepare fiber compositions change significantly under an intense thermal action (in particular, their strength decreases or increases) [9–12].

The objective of the present study is to examine the effect of temperature sensitivity and inelastic behavior of phase materials of the composition on the bearing capacity of structures in RR.

1. System of Resolving Equations and Boundary Conditions. A complete closed system of resolving equations of the RR problem, which describes, in the Cartesian coordinate system x_1Ox_2, the behavior of plane thermoelastic and thermoplastic structures statically loaded in their planes and reinforced by two families of uniformly stressed fibers (the binder and fiber materials are assumed to be isotropic, and the behavior of the binder is described by the deformation theory of plasticity [13]), includes the equations of equilibrium

$$(-1)^i \sum_k \sigma_k \omega_k l_{kj} \partial_k(\psi_k) + B_i(\boldsymbol{u}, \boldsymbol{\omega}, \varepsilon_0, \theta) = -(1-\Omega)F_i - \sum_k \omega_k F_{ki}$$

$$(j = 3 - i, \qquad i = 1, 2), \tag{1.1}$$

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written in displacements [14], the conditions of constant cross sections of the fibers

$$(\omega_k \cos\psi_k)_{,1} + (\omega_k \sin\psi_k)_{,2} = 0 \qquad (k = 1, 2), \tag{1.2}$$

the conditions of uniformly stressed reinforcement

$$\begin{gathered}\sigma_k = f_k(\varepsilon_k, \theta) = \text{const},\\ \partial_k(u_1)\cos\psi_k + \partial_k(u_2)\sin\psi_k - \alpha_k(\theta)\theta = \varepsilon_k(\theta) = f_k^{-1}(\sigma_k, \theta) \qquad (k = 1, 2),\end{gathered} \tag{1.3}$$

and the equation of plane stationary heat-conduction problem

$$\begin{gathered}(\Lambda_{11}(\theta)\theta_{,1} + \Lambda_{12}(\theta)\theta_{,2})_{,1} + (\Lambda_{21}(\theta)\theta_{,1} + \Lambda_{22}(\theta)\theta_{,2})_{,2} + 2\mu(\theta)(\theta_\infty - \theta)/h\\ = -(1-\Omega)Q - \sum_k \omega_k Q_k, \qquad \theta = T - T_0, \quad \theta_\infty = T_\infty - T_0.\end{gathered} \tag{1.4}$$

Here

$$B_i(\boldsymbol{u}, \boldsymbol{\omega}, \varepsilon_0, \theta) = a[g(\varepsilon_u, \theta)(u_{i,i} - \varepsilon_0) + 3K(\theta)(\varepsilon_0 - \alpha(\theta)\theta)]_{,i} + 0.5a[g(\varepsilon_u, \theta)(u_{i,j} + u_{j,i})]_{,j}, \ j = 3 - i, \ i = 1, 2; \tag{1.5}$$

$$\partial_k(\,\cdot\,) = l_{k1}\frac{\partial(\,\cdot\,)}{\partial x_1} + l_{k2}\frac{\partial(\,\cdot\,)}{\partial x_2}, \quad l_{k1} = \cos\psi_k, \quad l_{k2} = \sin\psi_k, \quad k = 1, 2; \tag{1.6}$$

$$\Omega = \sum_k \omega_k, \qquad \boldsymbol{\omega} = \{\omega_1, \omega_2\}, \qquad \boldsymbol{u} = \{u_1, u_2\}; \tag{1.7}$$

$$\begin{gathered}\Lambda_{ij}(\theta) = \frac{1}{\Omega}\sum_k \omega_k \Big\{[\Omega(\lambda_k(\theta) - \lambda(\theta)) + \lambda(\theta)]l_{ki}l_{kj} + \frac{(-1)^{i+j} l_{ks} l_{kr} \lambda_k(\theta)\lambda(\theta)}{\Omega(\lambda(\theta) - \lambda_k(\theta)) + \lambda_k(\theta)}\Big\},\\ s = 3 - i, \quad r = 3 - j, \quad i, j = 1, 2;\end{gathered} \tag{1.8}$$

$$\begin{gathered}g(\varepsilon_u, \theta) = \frac{2\sigma_u(\varepsilon_u, \theta)}{3\varepsilon_u}, \qquad K(\theta) = \frac{E(\theta)}{3(1 - 2\nu(\theta))}, \qquad \varepsilon_0 = \frac{\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}}{3},\\ \varepsilon_u = (\sqrt{2}/3)\sqrt{(\varepsilon_{11} - \varepsilon_{22})^2 + (2\varepsilon_{22} + \varepsilon_{11} - 3\varepsilon_0)^2 + (3\varepsilon_0 - 2\varepsilon_{11} - \varepsilon_{22})^2 + 6\varepsilon_{12}^2},\\ \varepsilon_{33} = 3\varepsilon_0 - \varepsilon_{11} - \varepsilon_{22}, \qquad \varepsilon_{ij} = (u_{i,j} + u_{j,i})/2, \qquad i,\, j = 1,\, 2;\end{gathered} \tag{1.9}$$

$$0 < a = \text{const} < 1. \tag{1.10}$$

On one part of the contour Γ_p, it is possible to set the static boundary conditions in displacements [14]

$$\begin{gathered}\sum_k \sigma_k \omega_k \cos^2(\psi_k - \beta) + D_{\rm n}(\boldsymbol{u}, \boldsymbol{\omega}, \varepsilon_0, \theta) = p_{\rm n},\\ \sum_k \sigma_k \omega_k \sin 2(\psi_k - \beta) + D_\tau(\boldsymbol{u}, \boldsymbol{\omega}, \varepsilon_0, \theta) = 2p_\tau, \qquad (x_1, x_2) \in \Gamma_p,\end{gathered} \tag{1.11}$$

on the other part Γ_u, one can set the kinematic conditions

$$u_i(\Gamma_u) = u_{i0}, \qquad i = 1, 2, \tag{1.12}$$

and on the entire contour $\Gamma = \Gamma_p \cup \Gamma_u$, it is possible to set the thermal conditions

$$\chi[(\Lambda_{11}(\theta)\theta_{,1} + \Lambda_{12}(\theta)\theta_{,2})n_1 + (\Lambda_{21}(\theta)\theta_{,1} + \Lambda_{22}(\theta)\theta_{,2})n_2 + q] + \gamma(\theta - \theta_0) = 0. \tag{1.13}$$

Here

$$\begin{gathered}D_{\rm n}(\boldsymbol{u}, \boldsymbol{\omega}, \varepsilon_0, \theta) = a\{g(\varepsilon_u, \theta)[u_{1,1}n_1^2 + u_{2,2}n_2^2 + (u_{1,2} + u_{2,1})n_1 n_2 - \varepsilon_0] + 3K(\theta)(\varepsilon_0 - \alpha(\theta)\theta)\},\\ D_\tau(\boldsymbol{u}, \boldsymbol{\omega}, \varepsilon_0, \theta) = ag(\varepsilon_u, \theta)[2(u_{2,2} - u_{1,1})n_1 n_2 + (u_{1,2} + u_{2,1})(n_1^2 - n_2^2)];\end{gathered} \tag{1.14}$$

$$n_1 = \cos\beta, \qquad n_2 = \sin\beta. \tag{1.15}$$

(It is possible to set conditions (1.11) and (1.12) on the entire contour Γ limiting the region G occupied by the structure in the planform.) On the part of the contour Γ_k in which the fibers of the kth family enter the structure, one has to specify the boundary conditions for reinforcement intensities:

$$\omega_k(\Gamma_k) = \omega_{0k}, \qquad k = 1, 2. \tag{1.16}$$

In solving the RR problem in the case of plane strain ($\varepsilon_{33} = 0$), one should take into account in operators (1.5) and (1.14) that

$$\varepsilon_0 = (\varepsilon_{11} + \varepsilon_{22})/3 = (u_{1,1} + u_{2,2})/3, \tag{1.17}$$

and in the case of the generalized plane stressed state (PSS), system (1.1)–(1.4) should be supplemented by the equation [14]

$$g(\varepsilon_u, \theta)(2\varepsilon_0 - u_{1,1} - u_{2,2}) + 3K(\theta)(\varepsilon_0 - \alpha(\theta)\theta) = 0, \qquad (x_1, x_2) \in G. \tag{1.18}$$

In addition, in the case of the linearly elastic behavior of the phase materials, the functions $g(\varepsilon_u, \theta)$, $f_k(\varepsilon_k, \theta)$, and $f_k^{-1}(\sigma_k, \theta)$ in relations (1.3), (1.5), (1.9), (1.14), (1.18) have the form

$$g(\varepsilon_u, \theta) = E(\theta)/(1 + \nu(\theta)), \qquad f_k(\varepsilon_k, \theta) = E_k(\theta)\varepsilon_k, \qquad f_k^{-1}(\sigma_k, \theta) = \sigma_k/E_k(\theta). \tag{1.19}$$

The solution of the RR problem should satisfy the physical constraints [3, 4, 14]

$$0 \leqslant \omega_k \quad (k = 1, 2), \qquad \Omega \leqslant 1 - a \quad (0 \leqslant a = \text{const} < 1) \tag{1.20}$$

and the strength constraints

$$\sigma_u(\varepsilon_u, \theta) \leqslant \sigma_\text{b}(\theta), \qquad -\sigma_k^-(\theta) \leqslant \sigma_k \leqslant \sigma_k^+(\theta), \qquad \sigma_\text{b} > 0, \qquad \sigma_k^\pm > 0, \quad k = 1, 2. \tag{1.21}$$

In equations and relations (1.1)–(1.21), F_i and F_{ki} are the components of specific volume loads acting on the binder and reinforcement of the kth family in the directions x_i, respectively, ω_k and ψ_k are the intensity and angle (counted from the direction x_1) of reinforcement by fibers of the kth family, σ_k and ε_k are the stress and mechanical strain of the fibers of the kth family (the tension–compression diagram $\sigma_k \sim \varepsilon_k$ can be asymmetric in the general case; its form depends on temperature and is determined by the function f_k), σ_u and ε_u are the stress and strain rates in the binder (the form of the diagram $\sigma_u \sim \varepsilon_u$ can be temperature-dependent), ε_{ij} and u_i are the strain and displacement components, ν is Poisson's ratio of the binder, E and E_k are the elasticity moduli of the binder and reinforcement of the kth family, respectively, a is the intensity of binder interlayers between the elementary reinforcement layers, α and α_k are the coefficients of linear thermal expansion of the binder and reinforcement of the kth family, λ and λ_k are the thermal conductivities of the binder and reinforcement of the kth family, θ is the structure-temperature difference in the working (T) and initial (T_0) states, θ_∞ is the temperature difference between the ambient medium T_∞ (on the side of the front surfaces of the structure) and T_0, μ is the coefficient of convective heat exchange between the binder and the ambient medium on the front surfaces of the plate (in the case of plane deformation, $\mu = 0$), $h = \text{const}$ is the plate thickness in the PSS, Q and Q_k are the powers of internal heat sources in the binder and fibers of the kth family, p_n and p_τ are the normal and tangential contour stresses, respectively, u_{i0} are the displacement components specified on the contour Γ_u, θ_0 is the temperature difference of the structural contour Γ in the working and initial states, q is the heat flux through the side surface of the structure, χ and γ are the toggle functions, which allow one to set different thermal conditions on Γ, β is the angle that defines the direction of the external normal to Γ, ω_{0k} are the values of the functions ω_k specified on the contour Γ_k, σ_b is the ultimate strength of the binder material, equal, for instance, to the yield point σ_y or to the time resistance σ_t, σ_k^- and σ_k^+ are the ultimate strengths of the fibers of the kth family under compression and tension, respectively (under the action of compressing loads, the fibers can lose stability; therefore, in the general case, $\sigma_k^- \neq \sigma_k^+$); summation is performed from 1 to 2; the subscript after the comma indicates partial differentiation with respect to the corresponding variable x_i. If the temperature sensitivity of substructural elements of the composition (TSSEC) is taken into account, their physicomechanical characteristics E, ν, K, α, λ, E_k, α_k, λ_k, μ, σ_y, σ_t, and $\sigma_k^\pm$ ($k = 1, 2$) depend on the structure temperature θ [9–12]; as a result, the effective thermal conductivities Λ_{ij} and the functions g and f_k, which characterize the strain diagrams of the phase materials, also depend on θ.

It is shown in [14] that the system of resolving equations (1.1)–(1.4), (1.18) [or (1.17)] is a quasilinear system of the mixed-composite type [15], which is closed relative to the unknown functions ψ_k, ω_k, u_k, θ, and ε_0 ($k = 1, 2$) and has two complex characteristics generated by the heat-conduction equation (1.4) and two real characteristics, which coincide with the trajectories of uniformly stressed fibers. The nonlinearity in the problem considered is caused by the “structural” nonlinearity (since the RR parameters ψ_k and ω_k are unknown functions) and by the physical nonlinearity (since the physicomechanical characteristics of the phases of the composition depend on the temperature θ and, in the case of the inelastic behavior of the phase materials, the functions $g(\varepsilon_u, \theta)$ and $f_k(\varepsilon_k, \theta)$ are nonlinearly expressed in terms of ε_u, ε_k, and θ whose values are determined from the problem solution). This imposes significant difficulties in the development of methods for solving the RR boundary-value problem.

2. Investigation of the Bearing Capacity of Plane Temperature-Sensitive Structures with Uniformly Stressed Reinforcement. The theory of the systems of quasilinear equations of the mixed-composite type has not been adequately developed [15], which does not allow analytical investigation of the properties of the solutions of the system of resolving equations of the RR problem in the general case. Still, an important class of solutions of this system can be identified and studied in more detail. Let us analyze the case of uniform deformation (UD) of a structure in the case where the total strains in the fibers of all families coincide with each other (e.g., the fibers are made of the same material) and the temperature field is uniform (this is possible for $Q = Q_k = 0$, $\mu = 0$, and $\theta(\Gamma) = \theta_0 = \text{const}$ [4]):

$$\varepsilon_{11} = \varepsilon_{22} = \varepsilon_k + \alpha_k\theta = \text{const}, \quad \varepsilon_{12} = 0, \quad \varepsilon_0 = \text{const}, \quad \theta = \text{const}, \quad k = 1, 2. \tag{2.1}$$

If conditions (2.1) are satisfied in RR structures, not only the fibers but also the binder are uniformly stressed [14]:

$$\sigma_{\mathrm{b},ii} = g(\varepsilon_u, \theta)(\varepsilon_{ii} - \varepsilon_0) + 3K(\varepsilon_0 - \alpha\theta) = \text{const}, \qquad \sigma_{\mathrm{b},12} = g(\varepsilon_u, \theta)\varepsilon_{12} = 0, \qquad i = 1, 2. \tag{2.2}$$

This allows us to eliminate the undesirable action of shear strains on reinforcement–binder cohesion and significantly increase crack resistance of the binder-matrix material.

If equalities (2.1) are satisfied, we have the operators $B_i = 0$ in the equilibrium equations (1.1) and the operators $D_{\mathrm{n}} = \text{const}$ and $D_\tau = 0$ in the static boundary conditions (1.11). For the inelastic behavior of the binder material, we have

$$\begin{gathered} D_{\mathrm{n}}(\boldsymbol{u}, \boldsymbol{\omega}, \varepsilon_0, \theta) = a[g(\varepsilon_u, \theta)(\varepsilon_1 + \alpha_1\theta - \varepsilon_0) + 3K(\varepsilon_0 - \alpha\theta)] = \text{const}, \\ \varepsilon_u = 2\sqrt{(\varepsilon_1 + \alpha_1\theta - \varepsilon_0)^2} = 2|\varepsilon_1 + \alpha_1\theta - \varepsilon_0| = \text{const} \end{gathered} \tag{2.3}$$

in the case of plane deformation, we obtain $\varepsilon_0 = 2(\varepsilon_1 + \alpha_1\theta)/3$ [see (1.17) and (2.1)], and in the case of PSS, ε_0 is determined from the equation [see (1.9), (1.18), and (2.1)]

$$3K(\varepsilon_0 - \alpha\theta) - 2g(\varepsilon_u, \theta)(\varepsilon_1 + \alpha_1\theta - \varepsilon_0) = 0. \tag{2.4}$$

In the case of the linearly elastic behavior of the binder material, by virtue of (1.19), we obtain

$$D_{\mathrm{n}}(\boldsymbol{u}, \boldsymbol{\omega}, \varepsilon_0, \theta) = Ea[\varepsilon_1 + (\alpha_1 - \alpha)\theta]/(1 - \nu) = \text{const}. \tag{2.5}$$

Since $B_i = 0$ ($i = 1, 2$), $D_{\mathrm{n}} = \text{const}$, and $D_\tau = 0$, the solution of the RR problem in the case of uniform deformation of the structure is constructed identically for both the elastic and inelastic behavior of the binder material. [In the case of the inelastic behavior of the binder material, in the case of PSS, one only have to solve preliminary Eq. (2.4) with respect to $\varepsilon_0 = \text{const}$]. The thermoelastic RR problem for plane structures under UD was considered in detail in [6]. All results obtained in [6] can be transposed to the case of the inelastic behavior of phase materials of the composition. In particular, to satisfy conditions (2.1), only two families of reinforcement should be inserted into the structure: in the absence of volume loads ($F_i = F_{ki} = 0$), the RR trajectories are straight lines, which is convenient for implementation of the corresponding projects. In the case of axisymmetric loading of annular plates under UD and $F_i = F_{ki} = 0$ ($i, k = 1, 2$), the solution of the RR problem can be obtained in an analytical form [6].

Let us analyze some solutions of the RR problem for plane structures under UD. Let an annular plate be limited by circumferences of radii r_0 and r_1 ($r_0 < r_1$). Both contours experience uniform normal loads $p_{\mathrm{n},0} = \text{const}$ and $p_{\mathrm{n},1} = \text{const}$ ($p_{\tau,0} = p_{\tau 1} = 0$), respectively, and there are no volume loads. Both families of fibers are made of the same material:

$$\sigma_1 = \sigma_2 = \text{const}, \quad \varepsilon_1 = \varepsilon_2, \quad E_1 = E_2, \quad \alpha_1 = \alpha_2, \quad f_1(\varepsilon, \theta) = f_2(\varepsilon, \theta). \tag{2.6}$$

TABLE 1

$T, ^\circ$C	MA2 magnesium alloy [11]						Boron fibers [10]			
	E, GPa	$\sigma_{0,2}$, MPa	$\sigma_{\rm t}$, MPa	δ	$\alpha \cdot 10^6$, K^{-1}	ν	E_1, GPa	$\sigma_{{\rm t},1}$, MPa	δ_1	$\alpha_1 \cdot 10^6$, K^{-1}
20	44.5	190	250	0.15	32.4	0.31	416.5	3150	0.002	2.4
150	41.2	98	167	0.22	33.1	0.33				

Since the form and loading of the structure possess axial symmetry, it is reasonable to seek an axisymmetric solution of the RR problem. In the axisymmetric case, in the polar coordinate system (r, φ), for $F_i = F_{ki} = 0$, the rectilinear RR trajectories are defined by the equations [6]

$$r \sin \tilde{\psi}_k(r) = C_k = \text{const} \qquad (k = 1, 2), \tag{2.7}$$

where $\tilde{\psi}_k = \psi_k - \varphi$ are the reinforcement angles counted from the direction of the polar radius r and C_k are constants to be determined, whose absolute values are equal to the distances from the origin to the reinforcement trajectory of the kth family. The reinforcement intensity ω_k is determined by the expression

$$r\omega_k(r) \cos \tilde{\psi}_k(r) = r_0 \omega_{0k} \cos \tilde{\psi}_k(r_0) = \text{const}, \qquad \omega_{0k} = \omega_k(r_0), \qquad k = 1, 2. \tag{2.8}$$

The constants C_k and ω_{0k} are determined from the static boundary conditions at the points $r = r_0, r_1$. In particular, if we consider only radially symmetric RR structures ($\tilde{\psi}_2 = -\tilde{\psi}_1$, $\omega_1 = \omega_2$, and $\omega_{01} = \omega_{02}$), then, with allowance for (2.6), we obtain [6]

$$\begin{gathered} C_k^2 = R_0(\omega_{01})(P_0 - \omega_{01})/[2\omega_{01}(P_0 - \omega_{01})], \qquad k = 1, 2, \\ 2\omega_{01} = [(r_1^2 P_1)^2 - (r_0^2 P_0)^2][r_0^2(r_1^2 - r_0^2)P_0]^{-1}, \qquad \omega_{01} = \omega_{02}, \end{gathered} \tag{2.9}$$

where

$$R_0(\omega_{01}) = r_0^2(2\omega_{01} - P_0), \qquad P_i = (p_{{\rm n},i} - D_{\rm n})/\sigma_1, \qquad i = 0, 1, \tag{2.10}$$

and the values of $D_{\rm n}$ are determined by expressions (2.3) or (2.5).

It follows from relations (2.7), (2.8) that the physical constraints (1.20) are satisfied for the entire structure if they are satisfied on the inner contour. This requirement yields the inequalities

$$0 \leqslant [(r_1^2 P_1)^2 - (r_0^2 P_0)^2][r_0^2(r_1^2 - r_0^2)P_0]^{-1} < 1 - a \qquad (a = \text{const}, \quad a < 1 - \Omega); \tag{2.11}$$

$$0 < \cos^2 \tilde{\psi}_1(r_0) = P_0/(2\omega_{01}) \leqslant 1, \tag{2.12}$$

which determine, with allowance for the dependences of P_0 and P_1 on $p_{{\rm n},0}$, $p_{{\rm n},1}$, and θ [see (2.10)], in the phase space $(p_{{\rm n},0}, p_{{\rm n},1}, \theta)$, the region of admissible thermoforce loading of the structure, at which the solution of the RR problem for annular structures under UD can exist for $\omega_{01} = \omega_{02}$.

By the example of solving the RR problem for an annular plate under UD, the efficiency of using inelastic projects with uniformly stressed reinforcement can be conveniently shown. We consider an annular plate limited by circumferences of radii r_0 and r_1 ($r_0/r_1 = 0.5$), which is made of the MA2 magnesium alloy and reinforced by two families of boron fibers. The physicomechanical characteristics of the phase materials are listed in Table 1. The distributed volume loads are ignored ($F_i = F_{ki} = 0$, where $i, k = 1, 2$), and the uniformly distributed normal stresses are set at the contours of the structure:

$$p_{{\rm n},0} = 0.45\sigma_{{\rm t},1} p, \qquad p_{{\rm n},1} = 0.25\sigma_{{\rm t},1} p, \qquad p_{\tau,0} = p_{\tau,1} = 0 \tag{2.13}$$

($\sigma_{{\rm t},1}$ is the stress equal to the time resistance of boron fibers and $p > 0$ is a loading parameter). The generalized PSS is formed in the plate.

First, we consider the case of the elastic behavior of the binder material (boron fibers behave as elastobrittle ones). We have to evaluate the limiting value of mechanical strain of reinforcement ε_1 for which the intensity of stresses in the binder under UD is equal to the yield point ($\sigma_{\rm y} = \sigma_{0,2}$). Equations (1.21) and (2.2) for $\sigma_{\rm b} = \sigma_{\rm y}$ yield the equation for the limiting value of ε_1:

$$\sigma_u = \sqrt{\sigma_{{\rm b},11}^2 - \sigma_{{\rm b},11}\sigma_{{\rm b},22} + \sigma_{{\rm b},22}^2} = E|\varepsilon_1 + (\alpha_1 - \alpha)\theta|/(1 - \nu) = \sigma_{\rm y}. \tag{2.14}$$

TABLE 2

p	$T = 20°$C ($\theta = 0°$C)		$T = 150°$C ($\theta = 130°$C)			
			Without allowance for TSSEC		With allowance for TSSEC	
	EB	IB	EB	IB	EB	IB
$p_{\min}$	0.361	0.368	0.361	0.3685	0.1865	0.193
$p_{\max}$	0.58	1.31	1.193	1.305	0.945	1.25

Note. The abbreviations EB and IB refer to the elastic and inelastic behavior of the binder, respectively.

We denote the solution of Eq. (2.14) as $\varepsilon_1^{\rm y}$:

$$\varepsilon_1^{\rm y} = (1-\nu)\sigma_{\rm y}/E + (\alpha - \alpha_1)\theta \qquad \text{for} \qquad \varepsilon_1^{\rm y} + (\alpha_1 - \alpha)\theta > 0,$$
$$\varepsilon_1^{\rm y} = -(1-\nu)\sigma_{\rm y}/E + (\alpha - \alpha_1)\theta \qquad \text{for} \qquad \varepsilon_1^{\rm y} + (\alpha_1 - \alpha)\theta < 0. \tag{2.15}$$

Since the loading parameter p in (2.13) is assumed to be positive, then, for $\theta = 0$, the value of $\varepsilon_1^{\rm y}$ is defined by the first equality in (2.15).

For the structure temperature in the initial state ($T_0 = 20°$C and $\theta = 0°$C), we have $\varepsilon_1^{\rm y} = 0.39\varepsilon_{\rm t,1}$ ($\varepsilon_{\rm t,1} = \sigma_{\rm t,1}/E_1$ is the limiting strain of boron fibers). This means that the stress in uniformly stressed reinforcement in the case of the elastic behavior of the binder material cannot exceed 39% of the ultimate strength $\sigma_{\rm t,1}$. Hence, in the case of the elastic behavior of the binder material, which was the MA2 alloy, the bearing capacity of boron fibers is used only partially.

Let us consider RR projects, in which the bearing capacity of reinforcement is used to the maximum extent, i.e., $\sigma_k = \sigma_{\rm t,1}$. In this case, we have $\varepsilon_1^{\rm y} < \varepsilon_k = \varepsilon_{\rm t,1}$ ($k = 1, 2$), and plastic strains arise in the binder at $\theta = 0°$C. To solve the inelastic RR problem, one has to solve first Eq. (2.4) with respect to ε_0; with allowance for the expressions for $g(\varepsilon_u, \theta)$ and ε_u [see (1.9) and (2.3)], this equation is transformed to

$$9K(\varepsilon_0 - \alpha\theta) - 2\,\text{sign}\,(\varepsilon_1 + \alpha_1\theta - \varepsilon_0)\sigma_u(\varepsilon_u, \theta) = 0. \tag{2.16}$$

The solution of Eq. (2.16) depends on the form of the dependence $\sigma_u(\varepsilon_u, \theta)$ for the binder material. We assume that the diagram $\sigma_u \sim \varepsilon_u$ has a sector with linear reinforcement. Then, the dependence $\sigma_u(\varepsilon_u, \theta)$ is determined by the relations [13]

$$\sigma_u(\varepsilon_u, \theta) = E_u(\theta)\varepsilon_u \qquad \text{for} \qquad 0 \leqslant \varepsilon_u \leqslant \varepsilon_{u,\rm y}(\theta),$$
$$\sigma_u(\varepsilon_u, \theta) = \sigma_{\rm y}(\theta) + E_{u,\rm y}(\theta)(\varepsilon_u - \varepsilon_{u,\rm y}(\theta)) \qquad \text{for} \qquad \varepsilon_{u,\rm y}(\theta) \leqslant \varepsilon_u \leqslant \varepsilon_{u,\rm t}(\theta), \tag{2.17}$$

where

$$E_u = 1.5E/(1+\nu), \qquad \varepsilon_{u,\rm y} = 2(1+\nu)\sigma_{\rm y}/(3E), \qquad \varepsilon_{u,\rm t} = \delta + 2(1+\nu)\sigma_{\rm t}/(3E),$$
$$E_{u,\rm y} = E(\sigma_{\rm t} - \sigma_{\rm y})/[E\delta + 2(1+\nu)(\sigma_{\rm t} - \sigma_{\rm y})/3]. \tag{2.18}$$

Substituting the second relation of (2.17) into (2.16) and taking into account the expression for ε_u (2.3) and (2.18), we obtain

$$9K(\varepsilon_0 - \alpha\theta) - 2\,\text{sign}\,(\varepsilon_1 + \alpha_1\theta - \varepsilon_0)(\sigma_{\rm y} - E_{u,\rm y}\varepsilon_{u,\rm y}) - 4E_{u,\rm y}(\varepsilon_1 + \alpha_1\theta - \varepsilon_0) = 0. \tag{2.19}$$

Equation (2.19) has two solutions:

$$\varepsilon_0 = [9K\alpha\theta + 4E_{u,\rm y}(\varepsilon_1 + \alpha_1\theta) \pm 2(\sigma_{\rm y} - E_{u,\rm y}\varepsilon_{u,\rm y})]/(9K + 4E_{u,\rm y}). \tag{2.20}$$

The choice of the sign in (2.20) depends on the sign of the inequality

$$\varepsilon_u = \pm 2(\varepsilon_1 + \alpha_1\theta - \varepsilon_0) > \varepsilon_{u,\rm y} > 0. \tag{2.21}$$

The minus and plus signs should be chosen in (2.20) and (2.21) in the case of fiber compression ($\varepsilon_1 < 0$) and extension ($\varepsilon_1 > 0$), respectively. [This choice of the solution of Eq. (2.19) remains valid for the composition considered in the realistic range of temperatures $-300°\text{C} \leqslant \theta \leqslant 1000°$C.]

For the value of ε_0 known from (2.20) and (2.21), using formulas (2.7)–(2.10), we can determine the axisymmetric RR structure for an annular plate under UD in the case of the inelastic behavior of the binder material.

Based on the above-described scheme, calculations were performed for the elastic and inelastic behavior of the binder material. It was assumed that the stressed state in the binder reached the yield point ($\sigma_u = \sigma_{\rm y}$) in

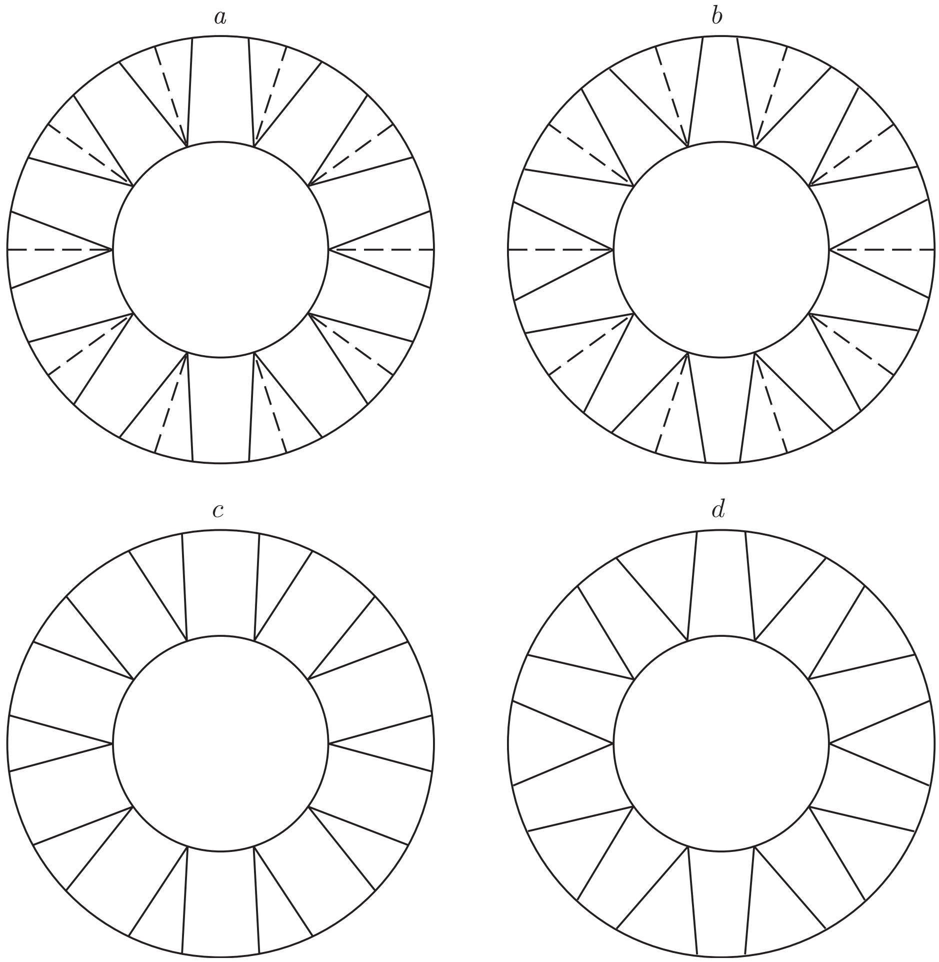

Fig. 1. RR structures of uniformly deformed annular plates under thermoforce loading in the absence of the thermal action (a–c) and under heating (d): (a) elastic behavior of the binder material; (b) inelastic behavior (the solid and dashed lines refer to $p = p_{\max}$ and $p = p_{\min}$, respectively); (c, d) elastic and inelastic behavior of the binder material ($p = 0.45$).

the case of the elastic behavior, and the stresses in the reinforcement reach the time resistance ($\sigma_1 = \sigma_{t,1}$) in the case of the inelastic behavior. It turned out that different values of the loading parameter p in (2.13) correspond to different RR projects in the cases of elastic and inelastic behavior of the binder material. Table 2 shows the minimum value ($p_{\min}$) and maximum value ($p_{\max}$) of the parameter p for which it is possible to obtain the solution of the RR problem with the above-described features of the stressed state in the phases of the composition. The values of $p_{\min}$ correspond to degeneration of the reinforcement structure into the radial structure ($\tilde{\psi}_1 = \tilde{\psi}_2 = 0$) [i.e., the sign of equality occurs in constraint (2.12)], and the values of $p_{\max}$ correspond to the limiting concentration of reinforcement on the inner contour r_0, i.e., $2\omega_{01} = 1 - a$ [see (2.9) and (2.11)]; we used $a = 0.3$ in the calculations; therefore, the limiting value of ω_{01} is 0.35.

Figure 1a and b shows the RR structures of the annular plate under UD in the case of the elastic and inelastic behavior of the binder material, respectively, at a temperature $T = 20°$C ($\theta = 0°$C) (the dashed lines indicate the RR structures corresponding to the minimum values of the loading parameter $p = p_{\min}$ and the solid lines, to the maximum values $p = p_{\max}$).

It follows from Table 2 that, for $\theta = 0°$C, the highest load that can be sustained by the RR structure with the inelastic behavior of the binder material is 2.26 times that with the elastic behavior. The reason is that the stress in reinforcement in the elastic project, as was mentioned above, does not exceed 39% of the ultimate strength $\sigma_{t,1}$, whereas the corresponding value in the inelastic project is $\sigma_{t,1}$. In addition, it follows from Table 2 that the solution of the RR problem can be obtained both in the elastic and inelastic cases for $0.368 \leqslant p \leqslant 0.58$ and $\theta = 0°$C. In this range of the values of p, it is reasonable to compare the amount of reinforcement used in the elastic and

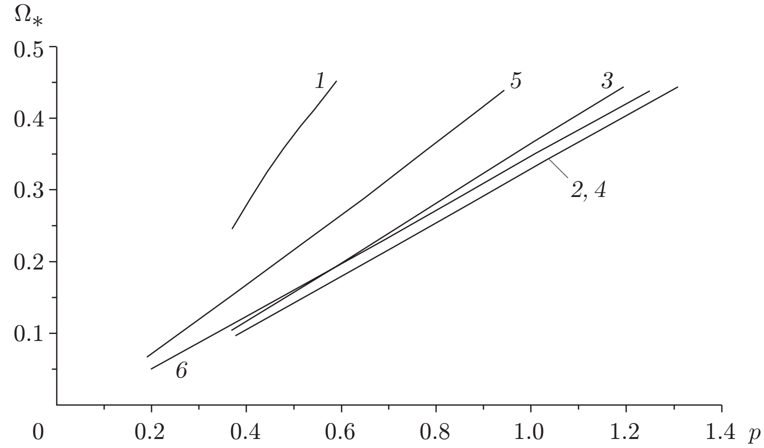

Fig. 2. Relative volume of fibers in uniformly deformed plates versus the loading parameter for $T = 20°$C (1, 2) and $T = 150°$C (3–6): elastic behavior of the binder material (curve 1), inelastic behavior (curve 2), elastic behavior (TSSEC ignored) (curve 3), inelastic behavior (TSSEC ignored) (curve 4), elastic behavior (TSSEC included) (curve 5), and inelastic behavior (TSSEC included) (6).

inelastic projects. The relative volume concentration of fibers Ω_* in the structure is determined by the formula

$$\Omega_* = \frac{1}{S_G} \sum_k \iint\limits_G \omega_k dx_1\, dx_2 = \frac{2}{S_G} \int\limits_0^{2\pi} \int\limits_{r_0}^{r_1} \omega_k r\, dr\, d\varphi, \qquad S_G = \pi(r_1^2 - r_0^2).$$

Figure 2 shows the dependences $\Omega_*(p)$ for elastic and inelastic RR projects for different values of temperature. Curves 1 and 2 are obtained for $T = 20°$C in the elastic and inelastic cases, respectively. A comparison of these curves in the interval $0.368 \leqslant p \leqslant 0.58$ shows that the overall consumption of reinforcement in the elastic RR project is more than twice greater than that in the inelastic project. Figure 1c shows the RR structure obtained for $p = 0.45$ and $\theta = 0°$C. Under such loading, the RR trajectories in the elastic and inelastic projects can be hardly distinguished visually, and Ω_* in the elastic project is 2.65 times greater than that in the inelastic project.

It should be noted that $p_{\min} > 0$ (see Table 2). RR projects can also be obtained, however, for $0 \leqslant p < p_{\min}$. Indeed, in determining $p_{\min}$ in the elastic case, it was assumed that $\sigma_u = \sigma_{\rm y}$; therefore, $p_{\min} > 0$. If the RR structure is obtained in the elastic case for a certain value $p = p_0$ within the interval $p_{\min} \leqslant p_0 \leqslant p_{\max}$, then, "fixing" this structure and varying p in the range $0 \leqslant p < p_0$, we obtain the plane problem of the linear theory of elasticity for an anisotropic medium. Then, the stress-strain state in the phases of the composition also changes proportionally to p (in particular, the stresses in reinforcement and binder are constant everywhere in G and proportional to p). Thus, we can also obtain RR projects for $0 \leqslant p < p_{\min}$, with $\sigma_u < \sigma_{\rm y}$. Obviously, with p varied in the interval $0 \leqslant p < p_0$, it is reasonable to use the elastic RR project corresponding to the value $p_0 = p_{\min}$, since this yields the smallest overall consumption of reinforcement (curve 1 in Fig. 2).

Let us study the influence of the thermal action on the bearing capacity of the structure considered. We assume that the plate is heated to a temperature $T = 150°$C ($\theta = 130°$C) and TSSEC is ignored, i.e., we perform calculations for the values of the physicomechanical characteristics of the phases, which are given in the first row of Table 1. The minimum ($p_{\min}$) and maximum ($p_{\max}$) values of the parameter p obtained in this case with the elastic and inelastic behavior of the binder material are given in the third and fourth columns of Table 2. A comparison of the values of $p_{\min}$ and $p_{\max}$ in the inelastic case in the absence ($T = 20°$C) and presence ($T = 150°$C) of the thermal action shows that the presence of the temperature field without allowance for TSSEC has almost no effect on the bearing capacity of the RR structure. Vice versa, in the case of the elastic behavior of the binder material, the value of $p_{\max}$ in the thermoelastic project ($T = 150°$C) is 2.06 times higher than the corresponding value in the elastic case ($T = 20°$C), though the values of $p_{\min}$ in these cases are identical. A drastic increase in the bearing capacity of the thermoelastic RR structure is explained by the more complete use of the bearing capacity of reinforcement. Indeed, the limiting mechanical strain $\varepsilon_1^{\rm y}$ determined by formula (2.15), for $\theta = 130°$C is $\varepsilon_1^{\rm y} = 0.905\varepsilon_{{\rm t},1}$. This means that the stresses in reinforcement of the thermoelastic project are 90.5% of the ultimate strength, i.e., are 2.32 times higher than those in the elastic structure, in which $\varepsilon_1^{\rm y} = 0.39\varepsilon_{{\rm t},1}$.

The dependences $\Omega_*(p)$ in the elastic and inelastic cases for $T = 150°$C are plotted in Fig. 2 by curves 3 and 4, respectively, and curve 4 almost coincides with curve 2. Curves 3 and 4 are located closer to each other than curves 1 and 2 obtained for $T = 20°$C. The reason is that the bearing capacity of fibers in the structure is used more completely in the thermoelastic case (curve 3) than in the elastic case (curve 1).

The neglect of TSSEC in the presence of the thermal action, however, can lead to significantly overestimated or underestimated calculation results, because the strength characteristics $\sigma_{0,2}$ and $\sigma_{\rm t}$ of the MA2 alloy drastically decrease with increasing temperature, and the bearing capacity of such a material almost vanishes at $T \approx 500°$C [11] [in particular, at $T = 150°$C, the yield point $\sigma_{\rm y} = \sigma_{0,2}$ is almost twice as low as that at $T = 20°$C, (see Table 1)]. Therefore, for the case $T = 150°$C, it is reasonable to perform an additional calculation with allowance for TSSEC (physicomechanical characteristics of the phases of the composition for this case are listed in the second row in Table 1). The resultant limiting values of the loading parameter $p_{\min}$ and $p_{\max}$ are given in the fifth and sixth columns of Table 2. A comparison of these values in the inelastic case for $T = 20°$C and $T = 150°$C shows that, in the presence of the thermal action and allowance for TSSEC, the lower limit of loading $p_{\min}$ decreases by a factor of 1.91 and the upper limit $p_{\max}$ decreases by 4.6%, i.e., the maximum bearing capacity of the inelastic structure as a whole remains almost the same as that at the temperature of the structure in the initial state, despite the drastic decrease in strength characteristics of the binder with increasing temperature. The reason is that the mechanical characteristics of boron fibers in the temperature range under consideration are independent of T (see Table 1), and the stresses in reinforcement both at $T = 20°$C and $T = 150°$C are equal to the ultimate strength ($\sigma_1 = \sigma_{{\rm t},1}$), whereas the binder is actually responsible for redistribution of loads over elementary fibers only. Therefore, worsening of mechanical characteristics of the binder with increasing temperature has almost no effect on the bearing capacity of the structure as a whole.

A comparison of the values of $p_{\min}$ in the elastic ($T = 20°$C) and thermoelastic ($T = 150°$C) cases shows that the lower limit of loading decreases by a factor of 1.94 under heating (with allowance for TSSEC). The upper limit $p_{\max}$ under heating increases by a factor of 1.63. The reason is that the bearing capacity of reinforcement in the thermoelastic project is used more completely than in the elastic case. Indeed, despite the drastic decrease in the yield point of the binder $\sigma_{\rm y}$ under heating, the limiting strain in reinforcement $\varepsilon_1^{\rm y}$ [see (2.15)] increases and reached $\varepsilon_1^{\rm y} = 0.745\varepsilon_{{\rm t},1}$ at $\theta = 130°$C, i.e., the stress in reinforcement is 74.5% of the ultimate strength $\sigma_{{\rm t},1}$ and 1.91 times greater than the corresponding value in the elastic project. [If the structure considered is cooled, the stresses in reinforcement decrease. Thus, at $T = -30°$C ($\theta = -50°$C) and physicomechanical characteristics given in the first row of Table 1, the stresses in reinforcement are 19.1% of the ultimate strength; for $\theta = -98.2°$C, the stresses in reinforcement of the thermoelastic structure are zero. In the latter case, the bearing capacity of the structure is determined by the binder properties only.]

The dependences $\Omega_*(p)$ in the elastic and inelastic cases for $T = 150°$C with allowance for TSSEC are plotted by curves 5 and 6 in Fig. 2, respectively. These curves are located closer to each other than curves 1 and 2 obtained at $T = 20°$C, since the stresses in reinforcement reach 74.5% of the ultimate strength in the thermoelastic case (curve 5) and 39% in the elastic case (curve 1), whereas the stresses in reinforcement in inelastic projects are equal to the time resistance.

It follows from Table 2 and Fig. 2 that the solution of the RR problem can be obtained for $T = 150°$C and $0.193 \leqslant p \leqslant 0.945$ both for the elastic and inelastic behavior of the binder material with allowance for TSSEC. Figure 1d shows the reinforcement structure obtained for $p = 0.45$ and $T = 150°$C with allowance for TSSEC. In this case, the reinforcement trajectories in the thermoelastic and inelastic projects can be hardly distinguished visually, and the total consumption of reinforcement in the thermoelastic project is higher than that in the inelastic project by 34.9%.

A comparison of the values of $p_{\min}$ and $p_{\max}$ obtained for $T = 150°$C with and without allowance for TSSEC shows that the neglect of temperature sensitivity leads to a twofold increase in the lowest value of the loading parameter $p_{\min}$ both for the elastic and inelastic behavior of the binder material; the upper limit $p_{\max}$ increases by 26.2% in the thermoelastic case and by 4.4% in the inelastic case.

Thus, based on the analysis performed, we can conclude that the use of RR structures in the case of the inelastic behavior of phase materials of the composition sometimes allow a severalfold increase in the bearing capacity of the plate as compared to the case of the elastic behavior of materials of all phases of the composition.

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